Lecture 8 | let us now perform the analogous treatment (with the one in the previous lecture) for the shuffle algebras that pertain to quantum loop groups.

To this end, let C be a symmetric Cartan matrix, let I be its set of rows/columns, and let $\mathbb{K} = \mathbb{Q}(q)$ (though $\mathbb{K} = \mathbb{C}$ and $q \in \mathbb{C}^*$ not a root of unity would also work)

Let $$\varphi_{ij} = \frac{1 - x q^{d_{ij}}}{(1-x)^{\delta_{ij}}} (-x)^{-\delta_{i<j}}$$, where $\begin{cases} d_{ii} = 2 \\ d_{ij} = d_{ji} \in \mathbb{Z}_{\leq 0} \text{ for } i \neq j \end{cases}$ are entries of C

some total order on I

Recall that the name of the game is to define an explicit subalgebra

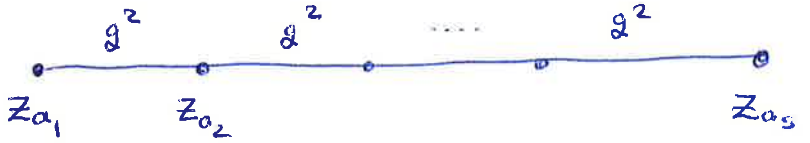
$$\mathcal{S}^+ \subset \mathcal{V}^+ = \bigoplus_{n = (n_i \geq 0)_{i \in I}} \mathbb{K}[z_{i1}^{\pm 1}, \dots, z_{in_i}^{\pm 1}]^{\text{sym}}_{i \in I} \quad \text{s.t.} \quad \langle \mathcal{S}^+, \operatorname{Ker}(\tilde{U}^- \to \mathcal{V}^-) \rangle = 0$$

more explicitly, we need to recall the pairing

$$\langle R^+, f_{i_1 d_1} \dots f_{i_n d_n} \rangle = \int_{|z_1| \ll \dots \ll |z_n|} \frac{R^+(z_1, \dots, z_n) z_1^{d_1} \dots z_n^{d_n}}{\prod_{1 \leq a < b \leq n} \varphi_{i_a i_b}\left(\frac{z_a}{z_b}\right)} \quad , \forall R \in \mathcal{S}^+$$

The goal is to move the contours toward the symmetric $|z_1| = \dots = |z_n|$, and to impose conditions on R^+ in order to kill "bad" poles along the way; as before, let us assume $|q| > 1$ is a complex number in order for the argument to run smoothly; because the poles of the integrand are all of the form $\{z_b = z_a q^{d_{i_a i_b}}\}_{a<b}$ and $d_{i_a i_b} \leq 0$ if $i_a \neq i_b$, then the only such poles which might hinder movement from $|z_b| \gg |z_a|$ to $|z_b| = |z_a|$ are $z_b = z_a q^2$ if $i_a = i_b$; so we will definitely encounter multivariate poles of the form $z_{a_s} = q^2 z_{a_{s-1}} = \dots = q^{2(s-1)} z_{a_1}$

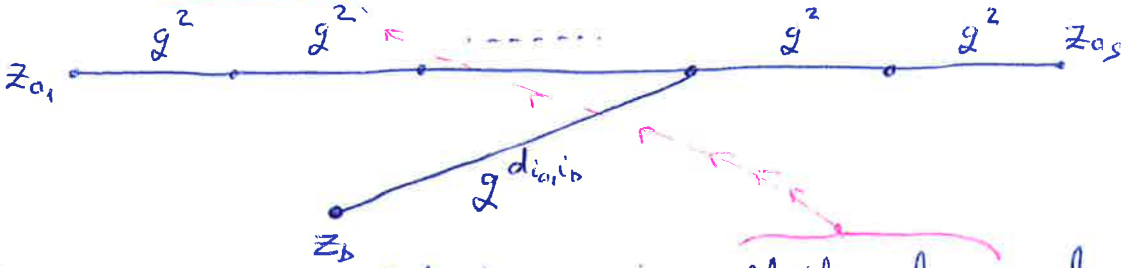
for various $a_1 < \dots < a_s$ such that $i_{a_1} = \dots = i_{a_s}$; now let's see what happens if we have some $b > a_s$ and we try to bring $|z_b|$ in from a very large number to somewhere around the size of $z_{a_1}, \dots, z_{a_s}$



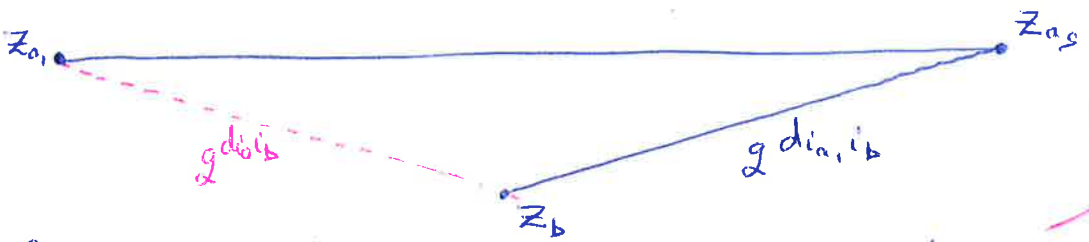
and z_b much larger

What kind of pole can we pick up as z_b passes over the geometric progression $z_{a_1}, z_{a_1} g^2, \ldots\ldots, z_{a_1} g^{2(s-1)}$? Because of $\zeta_y(x)$, we can have two types of residues
$$\begin{cases} z_b = z_{a_1} g^{2s} & \text{if } i_{a_1} = \ldots = i_{a_s} = i_b \quad (\text{extending the geometric progression}) \\ z_b = z_{a_1} g^{2x + d_{i_a, i_b}} & \text{if } x \in \{0, 1, \ldots, k-1\}, \text{ and } i_{a_1} = \ldots = i_{a_s} \neq i_b \end{cases}$$

i.e.

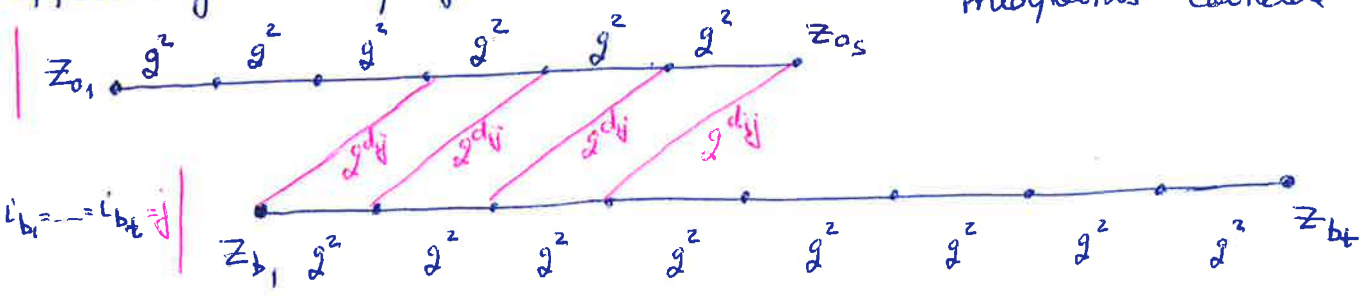


It would not be reasonable to require all the above poles to be annihilated by R^+; the corresponding set of R^{+}'s would not be closed under shuffle product $*$; however, you can require half of the poles above to be annihilated by R^+, and this would allow z_b to go as much as halfway between z_{a_1} and z_{a_s}



Feigin–Odesskii's "classic" wheel conditions

indeed, $\left\{ R^+ \text{ which vanishes when } z_{i_1} = \frac{z_{i_2}}{g^2} = \ldots\ldots = \frac{z_{i, 1+d_{ij}}}{g^{-2d_{ij}}}, \ z_{i_1} = z_{j_1} g^{d_{ij}} \right\} \forall i \neq j$ does indeed determine a subalgebra of $\mathcal{V}^+$ with respect to shuffle product; however, this is not enough for our argument to run through, because now the symmetric choice of contours is when all the geometric progressions have the same midpoint; so we have to deal with the possibility of two different geometric progressions which must be moved until their midpoints coincide

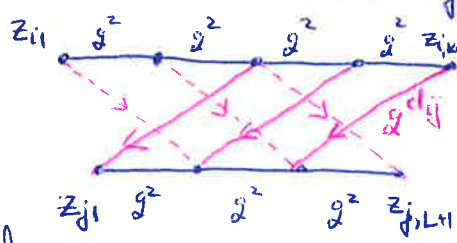


The pink lines in the previous picture is the number of poles in $\dfrac{R(z_1,\dots,z_n)\cdot z_1^{d_1}\cdots z_n^{d_n}}{\prod_{1\le a<b\le n}\varphi_{i_a i_b}\left(\frac{z_a}{z_b}\right)}$ that must be overcome by the numerator of R^+.

<u>Def</u>. let $\mathcal{S}^+ = \left\{ \begin{matrix} R^+ \text{ which is divisible by } (x-y)^m \text{ when we specialize} \\ z_{i1}=xq^{\frac{-k}{2}},\dots, z_{i,k+1}=xq^{k}, z_{j1}=yq^{-L}, \dots z_{j,L+1}=yq^{L} \end{matrix} \right\}_{\forall i\neq j,\ \forall k+L=-d_{ij}} \subset \mathcal{V}^+$

$m = \#$ pink pointing arrows in

$= \#$ pink pointing arrows in



The argument on the previous page leads to a formula

$$\langle R^+, f_{i_1 d_1}\dots f_{i_n d_n}\rangle = \sum_{\substack{\text{partitions of } \{1,\dots,n\} \\ \text{into } \{a_1<\dots<a_s\} \text{ s.t. } i_{a_1}=\dots=i_{a_s} \\ \{b_1<\dots<b_t\} \text{ s.t. } i_{b_1}=\dots=i_{b_t} \\ \dots\dots\dots}} \int_{|Y_a|=|Y_b|=\dots} \operatorname{Res}_{\substack{z_{a_1}=Y_a q^{-s+1},\dots, z_{a_s}=Y_a q^{s-1} \\ z_{b_1}=Y_b q^{-t+1},\dots, z_{b_t}=Y_b q^{t-1} \\ \dots\dots\dots}} \frac{R^+(z_1,\dots,z_n)R^-(z_1,\dots,z_n)}{\prod_{1\le a\neq b\le n}\varphi_{i_a i_b}\left(\frac{z_a}{z_b}\right)}$$

$\forall R^+ \in \mathcal{S}^+$

and so we have $\langle \mathcal{S}^+, K^-\rangle = 0 \Rightarrow$ pairing descends to $\mathcal{S}^+\otimes \mathring{\mathcal{S}}^- \longrightarrow \mathbb{K}$.

$\Downarrow$

<u>Prop</u>: $\mathcal{S}^+$ is an algebra $\Longrightarrow \mathcal{S}^+ \supseteq \mathring{\mathcal{S}}^+ \Longrightarrow \mathcal{S}^+ = \mathring{\mathcal{S}}^+$ and so we obtain a non-degenerate pairing $\mathcal{S}^+ \otimes \mathcal{S}^- \to \mathbb{K}$ in both arguments.

$$\begin{matrix} \mathcal{S}^+ & \otimes & \mathcal{S}^- \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ U^+ & \otimes & U^- \end{matrix} \to \mathbb{K}$$

Note: I do not know how to define $\mathcal{S}^+$ and run the above program if C is just a symmetrizable Cartan matrix instead of a symmetric one; the reason is that in the symmetrizable case, geometric progressions $\overset{q^{d_{ii}}}{\bullet\!-\!\bullet}\cdots\overset{q^{d_{ii}}}{\bullet\!-\!\bullet}$ might have different ratios, and complicated poles appear.

In the end, we descended the pairing and the linear conditions that cut out $\mathcal{S}^+ \subset \mathcal{V}^+$ are precisely dual to generators of the kernel of $\tilde{U}^- \twoheadrightarrow U^-$; as the former are explicit, let us work out the latter

$$\begin{tikzcd} \mathcal{V}^+ \arrow[r, phantom, "\otimes"] & \tilde{U}^- \arrow[d, two heads] \arrow[dr] & \\ \mathcal{S}^+ \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\otimes"] & U^- \arrow[r] & \mathbb{K} \end{tikzcd}$$

One can work out generators of $\tilde{U}^- \longrightarrow U^-$ in both cases we studied

thus producing a complete generators and relations presentation of $U^\pm$; but to keep things simple, let us only do this in the case of the intersection of these two examples, namely the situation in red: these two cases are actually equivalent, i.e. they produce isomorphic $\mathcal{S}^\pm$ and $U^\pm$

quantum loop groups associated to quivers

$\cup$

quivers with no loops or multiple edges, i.e. $|\vec{ij}| + |\vec{ji}| \leq 1, \forall i, j$

$\Longleftrightarrow$

quantum loop groups for symmetric Cartan matrices

$\cup$

simply-laced Cartan matrices $d_{ij} \in \{2, -1, 0\}\ \forall i,$

So let's take
$$\zeta_{ij}(x) = \begin{cases} \dfrac{1 - xq^2}{1-x} & \text{if } i = j \ (\text{i.e. } d_{ij} = 2) \\ (1 - xq^{-1})(-x)^{-\delta_{i<j}} & \text{if } d_{ij} = -1 \ (\text{we will indicate this as } i - j) \\ 1 & \text{if } d_{ij} = 0 \end{cases}$$

(note: we previously had $\zeta_{ij}(x) = (1-x)(-x)^{-\delta_{i<j}}$ if $d_{ij} = 0$; however, it's an easy exercise to show that multiplication by $\prod_{\substack{d_{ij}=0 \\ a,b}} (z_{ia} - z_{jb})$ yields an isomorphism between the shuffle algebras defined w.r.t. the red ζ and the one w.r.t. blue ζ)

Recall that in this case, we have
$$\mathcal{S}^+ = \left\{ R^+ \text{ which vanishes when we set } \left.\begin{matrix} z_{i2} = z_{i1} q^2 \\ z_{j1} = z_{i1} q \end{matrix} \right\} \forall i - j \right\} \subset \mathcal{V}^+$$

We must find elements $(\phi_{ij})_{i-j} \in \tilde{U}^-$ s.t. for any $R^+ \in \mathcal{V}^+$ we have
$$\langle R^+, \phi_{ij} \rangle = 0 \iff R^+ \Big|_{\substack{z_{i2} = z_{i1} q^2 \\ z_{j1} = z_{i1} q}} = 0 \quad (*)$$

Then it will follow on general grounds that $U^- = \tilde{U}^- / (\phi_{ij})_{\forall i-}$

Prop: Consider the following formal series of elements of $\tilde{U}^-$
$$\phi_{ij} = \left(\frac{y x_2}{q x_1} - \frac{x_2}{q^2}\right) f_i(x_1) f_i(x_2) f_j(y) + \left(\frac{x_1}{q} - x_2 q\right) f_i(x_2) f_j(y) f_i(x_1) + \left(\frac{y}{q} - x_2\right) f_j(y) f_i(x_1) f_i$$

Then relation $(*)$ holds (in fact, it suffices to consider a single coefficient of ϕ_{ij} of any given homogeneous degree, instead of the whole series).

Proof: let's assume WLOG that $i < j$, so $\zeta_{ij}(x) = q^{-1} - x^{-1}$, $\zeta_{ji}(x) = 1 - xq^{-1}$; we will only consider in the subsequent argument $R^+ = R^+(z_{i1}, z_{i2}, z_{j1})$; if R^+ has more variables than the above three, they simply go along for the ride

By definition, we have (in what follows, $\Big|_{x_1<<x_2<<Y}$ means "expand power series as $x_1<<x_2<<Y$")

$$\langle R^+, f_i(x_1) f_i(x_2) f_j(Y)\rangle = \frac{R^+(x_1,x_2,Y)(x_1-x_2)}{(x_1 q^2 - x_2)\left(\frac{1}{q}-\frac{Y}{x_1}\right)\left(\frac{1}{q}-\frac{Y}{x_2}\right)}\Bigg|_{|x_1|<<|x_2|<<|Y|}$$

$$\langle R^+, f_i(x_2) f_j(Y) f_i(x_1)\rangle = \frac{R^+(x_1,x_2,Y)(x_2-x_1)}{(x_2 q^2 - x_1)\left(\frac{1}{q}-\frac{Y}{x_2}\right)\left(1-\frac{Y}{x_1 q}\right)}\Bigg|_{|x_2|<<|Y|<<|x_1|}$$

$$\langle R^+, f_j(Y) f_i(x_1) f_i(x_2)\rangle = \frac{R^+(x_1,x_2,Y)(x_1-x_2)}{(x_1 q^2 - x_2)\left(1-\frac{Y}{x_1 q}\right)\left(1-\frac{Y}{x_2 q}\right)}\Bigg|_{|Y|<<|x_1|<<|x_2|}$$

$$\Rightarrow \langle R^+, \phi_{ij}\rangle = R^+(x_1,x_2,Y)(x_1-x_2)\underbrace{\left(\frac{1}{\left(1-\frac{x_1q^2}{x_2}\right)\left(1-\frac{Yq}{x_2}\right)}\Bigg|_{|x_1|<<|x_2|<<Y} + \frac{-1}{\left(1-\frac{Yq}{x_2}\right)\left(1-\frac{Y}{x_1q}\right)}\Bigg|_{|x_2|<<|Y|<<|x_1|} + \frac{1}{\left(1-\frac{x_1q^2}{x_2}\right)\left(1-\frac{Y}{x_1q}\right)}\Bigg|_{|Y|<<|x_1|<<|x_2|}\right)}_{=\sum_{k,l\in\mathbb{Z}}\left(\frac{Y}{x_1q}\right)^k\left(\frac{x_1q^2}{x_2}\right)^l}$$

Then note that $F(\alpha,\beta)\sum_{k,l\in\mathbb{Z}}\alpha^k\beta^l = 0$ as formal series $\Leftrightarrow F(1,1)=0$

We conclude that $\langle R^+,\phi_{ij}\rangle = 0 \Leftrightarrow R^+(x_1,x_2,Y)(x_1-x_2)\Big|_{\substack{Y=x_1q\\ x_2=x_1q^2}} \Leftrightarrow R^+(s,sq^2,sq)=0$

With this in mind, we conclude the definition of the quantum loop group as

$$U = U^+ \otimes \mathbb{K}[\varphi^{\pm}_{i,0},\varphi^{\pm}_{i,1},\varphi^{\pm}_{i,2},\dots]_{i\in I}\otimes U^-$$

where $U^- = \mathbb{K}\langle f_{i,d}\rangle_{i\in I, d\in\mathbb{Z}} \Big/$

$f_i(x)f_j(Y)(xq^2-Y) = f_j(Y)f_i(x)(x-Yq^2)$ if $i=j$

$f_i(x)f_j(Y)(x-Yq) = f_j(Y)f_i(x)(xq-Y)$ if $i-j$

$f_i(x)f_j(Y) = f_j(Y)f_i(x)$ if $i\not- j$ and $i\neq j$

$\phi_{ij}=0$ for all $i-j$

and let $U^+ = U^{-,op}$ with generators denoted by $e_{i,d}$ instead of $f_{i,d}$